On the minimal resolution conjecture (MRC) for points in general position in the projective space.

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Abstract

Let $S = \{P_1, \ldots, P_s\} \subset \mathbb{P}^n$ be a set of points in general position and denote by $R = [x_0, x_1, \ldots, x_n]$ the homogeneous coordinate ring of $\mathbb{P}^n$. Then the homogeneous ideal I_S of these points has a minimal graded free resolution of the form;

$$0 \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I_S \longrightarrow 0,$$

where $F_p = R(-d-p)^{a_{p-1}} \bigoplus R(-d-p-1)^{b_p}$, d is the smallest integer satisfying $s \leq \mathrm{h}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ and $\binom{d+n-1}{n} \leq s < \binom{d+4}{n}$, with $a_p b_p = 0$ for $p = 0, 1, 2 \ldots, n-2$. Proving the existence of a resolution of the above form is the same as showing that the Betti numbers a_i and b_i satisfy $a_i b_i = 0$. This is achieved by showing that the map below is of maximal rank using the method of Horace.

$$\mathrm{H}^0\left(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+1+p)\right) \longrightarrow \bigoplus_{i=1}^{s} \Omega^{p+1}_{\mathbb{P}^n}(d+1+p)|_{P_i}.$$

Key words: Betti numbers, maximal rank, method of Horace, minimal free resolution.

Overview

1. Introduction
2. Free resolutions and the Minimal resolution conjecture
3. Method of Horace
4. Conclusion

Introduction

Given a system of linear homogeneous equations of rank m in n variables over K.

Fact: If K is a field;

- Solving this system is equivalent to finding the kernel of a linear transformation $T : K^m \mapsto K^n$.
- There are $n - m$ linearly independent solutions.

Question

What if K is a polynomial ring $R = k[x_0, \ldots, x_n]$, where k is a field?

- The system may not have a set of linearly independent solutions from which all solutions can be obtained by taking R-linear combinations.

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- The system may not have a set of linearly independent solutions from which all solutions can be obtained by taking R-linear combinations.

Introduction Cont.

- The system can be viewed as a map of free modules over R, say $\delta_1 : F_1 \mapsto F_0$, and the process of finding a solution to the system yields a sequence of R-modules

$$\cdots \longrightarrow F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0$$

  called a free resolution of cokernel of δ_1.
- The resolution above is minimal if at every stage we choose a minimal set of generators for F_i.

remark

- Minimal free resolution contain information for important invariants like the Hilbert polynomial.
- The Castelnuovo-Mumford regularity, which gives a solution to the interpolation problem, is for the graded case given as the regularity of the minimal graded free resolution.

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Properties of minimal free resolutions for finitely generated graded modules

Let M be a finitely generated R-module;

1. The length of the minimal free resolution of M is $m \leq n+1$, the number of variables in R.

$$0 \longrightarrow F_m \xrightarrow{\delta_m} F_{m-1} \xrightarrow{\delta_{m-1}} F_{m-2} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0$$

2. The minimal free resolution above is unique up to isomorphism.
3. The differentials are each expressible as a matrix with entries in $\mathfrak{m}$ where $\mathfrak{m}$ is the homogeneous maximal ideal of R.
4. Each of the modules F_i are expressible as $F_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}}$.
5. Any minimal set of homogeneous generators of F_i contains exactly $dim_A Tor_i^R(A, M)_j$ generators of degree j, where A is the residue field $A \cong R/\mathfrak{m}$.
6. For s distinct points in $\mathbb{P}^n$ (where $\binom{d+n-1}{n} \leq s < \binom{d+n}{s}$) imposing independent conditions on forms of degree d, these torsional groups vanish for all i except for $i = d+1$ and $i = d+i+1$.

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Minimal resolution conjecture

Conjecture 2.1 (The minimal resolution conjecture (Lorenzini 1987))

Given a general set of points $S = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$, with $s \geq n+1$, then the homogeneous ideal of the sub-scheme of the union of these points, $I_S \subset R = k[x_0, \dots, x_n]$, k an algebraically closed field and R the homogeneous coordinate ring of $\mathbb{P}^n$, has the following expected form:

$$0 \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_0 \longrightarrow I_S \longrightarrow 0,$$

where $F_p = R(-d-p)^{a_{p-1}} \bigoplus R(-d-p-1)^{b_p}$, with $a_p b_p = 0$ for $0 \leq p \leq n-2$.

For s points in general position;

1. There exists an integer $s(n)$ such that for $s \geq s(n)$ there exist a non-empty Zariski open set $U \subset (\mathbb{P}^n)^s$ such that for all $(P_1, \dots, P_s) \in U$, for all $0 \leq p \leq n$ and for all $l \in \mathbb{Z}$ the evaluation map $\mathrm{H}^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(l)) \longrightarrow \bigoplus_{i=1}^s \Omega^p_{\mathbb{P}^n}(l)|_{P_i}$ is of maximal rank.
2. If $M = R/I_S$ and p_0 is the largest integer for which $s \leq \mathrm{h}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l - p_0))$, then $Tor^p_R(M, K)_l$ is non zero for exactly one of the values of p, namely $p = p_0$ or $p_0 + 1$ depending on whether the map $\mathrm{H}^0(\mathbb{P}^n, \Omega^{p_0}_{\mathbb{P}^n}(l)) \longrightarrow \mathrm{H}^0(S, \Omega^{p_0}_{\mathbb{P}^n}(l)|_S)$ is surjective or injective.
3. The homogenous ideal I_S has a resolution of the form

$$0 \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_0 \longrightarrow I_S \longrightarrow 0,$$

with $F_p = R(-p-l')^{a_p} \bigoplus R(-p-l'+1)^{b_p}$, and l' is the smallest integer such that $s \leq \mathrm{h}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l'))$ and

$$a_p = \max\{0, \mathrm{h}^0(\mathbb{P}^n, \Omega^{p-1}_{\mathbb{P}^n}(l'+p-1)) - \mathrm{rk}(\Omega^{p-1}_{\mathbb{P}^n}(l'+p-1))s\},$$

$$b_p = \max\{0, \mathrm{rk}(\Omega^p_{\mathbb{P}^n}(l'+p))s - \mathrm{h}^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(l'+p))\}.$$

Method of Horace

To develop a setting in which we will apply the method of Horace, consider a smooth projective variety X, and a non-singular divisor, X', of X. Let

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0 \quad (1)$$

and

$$0 \longrightarrow \mathcal{F}(-X) \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}'' \longrightarrow 0. \quad (2)$$

be exact sequences of coherent sheaves on X, where $\mathcal{E} = elem_{\mathcal{F}'}(F)$ and $\mathcal{F}(-X) = elem_{\mathcal{F}''}(E)$. If $\mathrm{H}^1(X, \mathcal{E}) = \mathrm{H}^1(X, \mathcal{F}(-X) = 0$, then applying the cohomology functor to the sequence 1 and 2 above, we get the following sequences;

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{E}) \longrightarrow \mathrm{H}^0(X, \mathcal{F}) \longrightarrow \mathrm{H}^0(X', \mathcal{F}') \longrightarrow 0 \quad (3)$$

and

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{F}(-X)) \longrightarrow \mathrm{H}^0(X, \mathcal{E}) \longrightarrow \mathrm{H}^0(X', \mathcal{F}'') \longrightarrow 0. \quad (4)$$

Using sequence 3, and 4 we we will formulate the inductive hypotheses for the method of Horace.

Method of Horace

Theorem 3.1 (Differential method of Horace)

Suppose we are given a surjective morphism of vector spaces, $\lambda : \mathrm{H}^0(X', \mathcal{F}') \longrightarrow L$ and suppose that there exist a point $Z' \in X'$ such that $\mathrm{H}^0(X', \mathcal{F}') \hookrightarrow L \oplus \mathcal{F}'|_{Z'}$ and suppose that $\mathrm{H}^1(X, \mathcal{E}) = 0$. Then there exist a quotient $\mathcal{E}(Z') \longrightarrow D(\lambda)$ with a kernel contained in $\mathcal{F}'(Z')$ of dimension $dim(D(\lambda)) = \mathrm{rk}(\mathcal{F}) - dim(ker\lambda)$ having the following property. Let $\mu : \mathrm{H}^0(X, \mathcal{F}) \longrightarrow M$ be a morphism of vector spaces, then there exist $Z \in X'$ such that if $\mathrm{H}^0(X, \mathcal{E}) \to M \oplus D(\lambda)$ is of maximal rank then $\mathrm{H}^0(X, \mathcal{F}) \longrightarrow M \oplus L \oplus \mathcal{F}(Z)$ is also of maximal rank.

The idea of the theorem is illustrated in the diagram below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0(X,\mathcal{E}) & \longrightarrow & \mathrm{H}^0(X,\mathcal{F}) & \longrightarrow & \mathrm{H}^0(X',\mathcal{F}') & \longrightarrow & 0 \\ & & \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\ 0 & \longrightarrow & M \oplus D(\lambda) & \longrightarrow & M \oplus L \oplus \mathcal{F}(Z) & \longrightarrow & L \oplus D'(\lambda)|_Z & \longrightarrow & 0 \end{array}$$

Key Point

The key point is that if the map α_3 is bijective, then α_2 will be bijective provided that α_1 is bijective.

To put the method of Horace in the context of the minimal resolution conjecture for $\mathbb{P}^n$, we begin by describing a suitable elementary transformation.

Proposition 3.2

There exist an elementary transformation of vector bundles on $\mathbb{P}^n$ comprising of the following exact sequences.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^{p+1}_{\mathbb{P}^n}(d+p) & = & \Omega^{p+1}_{\mathbb{P}^n}(d+p) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus \binom{n}{p+1}} & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^n}(d+p+1) & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^{n-1}}(d+p+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^{p}_{\mathbb{P}^{n-1}}(d+p) & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^n|\mathbb{P}^{n-1}}(d+p+1) & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^{n-1}}(d+p+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{5}
$$

Remark 3.3

The middle sequence (respectively the left most sequence) correspond to the sequence 1 (respectively sequence 2).

Strategy

1. Fix p in the diagram above and twist the sheaves by $d-1$.
2. Formulate inductive hypotheses.
3. Formulate the variant methods using theorem 3.1.
4. Use the variant methods to prove the inductive hypotheses.

Conclusion

Recall that the minimal resolution conjecture assert that if $S = \{P_1, \ldots, P_s\} \subset \mathbb{P}^n$ is a set of points in general position, where $s \geq n+1$, and R denote the homogeneous coordinate ring of $\mathbb{P}^n$, then the homogeneous ideal I_S of the subscheme of union of these points has a minimal graded free resolution of the form;

$$0 \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I_S \longrightarrow 0,$$

where $F_p = R(-d-p)^{a_{p-1}} \bigoplus R(-d-p-1)^{b_p}$, d being the smallest integer satisfying $s \leq \mathrm{h}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, $\binom{d+n-1}{n} \leq s \leq \binom{d+n}{n}$, with the Betti numbers a_p and b_p defined as;

$$a_p = \max\{0, \mathrm{h}^0(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)) - \mathrm{rk}(\Omega^{p+1}_{\mathbb{P}^n}(d+p+1))s\}$$

$$b_p = \max\{0, \mathrm{rk}(\Omega^{p+1}_{\mathbb{P}^n}(d+p+1))s - \mathrm{h}^0(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p+1))\}$$

satisfying $a_p b_p = 0$ for $p = 0, 1, 2, \ldots n-2$.
The main task is to prove that the evaluation map below is of maximal rank,

$$\mathrm{H}^0(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^4}(d+p+1)) \longrightarrow \bigoplus_{i=1}^{s} \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)_{|P_i}.$$

Proving bijectivity of the map above is the same as showing that $a_p = b_p = 0$.
If the number of points, $s < \mathrm{h}^0\left(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right)$, then the evaluation map above is injective. In this case $a_p = 0$ and $a_p b_p = 0$ as required.
If the number of points, $s > \mathrm{h}^0\left(\mathbb{P}^n, \Omega^{P+1}_{\mathbb{P}^n}(d+p+1)\right)$, then the evaluation map above is surjective. In this case $b_p = 0$ and $a_p b_p = 0$ as required.

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Thank you